

MATRIX HERMITE–HADAMARD TYPE INEQUALITIES

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ABSTRACT. We present several matrix and operator inequalities of Hermite–Hadamard type. We first establish a majorization version for monotone convex functions on matrices. We then utilize the Mond–Pecaric method to get an operator version for convex functions. We also present some applications. Finally we obtain an Hermite–Hadamard inequality for operator convex functions, positive linear maps and operators acting on Hilbert spaces.

1. INTRODUCTION

The following fundamental inequality, which was first published by Hermite in 1883 in an elementary journal and independently proved in 1893 by Hadamard in [11], is well known as the Hermite–Hadamard inequality in the literature:

$$(y - x)f\left(\frac{x + y}{2}\right) \leq \int_0^1 f(t) dt \leq (y - x)\frac{f(x) + f(y)}{2}, \quad (1.1)$$

where f is a convex function on an interval $[x, y]$. It provides a two-sided estimate of the mean value of a convex function. If f is convex on a segment $[a, b]$ of a linear space, one can easily observe that (1.1) is equivalent to the following double inequality:

$$f\left(\frac{a + b}{2}\right) \leq \int_0^1 f(ta + (1 - t)b) dt \leq \frac{f(a) + f(b)}{2}. \quad (1.2)$$

The Hermite–Hadamard inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [13]. During the last decades several interesting generalizations, special cases and formulations of this significant inequality for some types of functions f and various frameworks have been obtained. It gives indeed a necessary and sufficient condition for a function f to be convex. We would like to refer the reader to [9, 3, 14, 20, 16, 1, 4, 12, 22, 7] and references therein for more information. In particular, Dragomir [8] very recently established an operator version of the inequality for the operator convex functions. In fact, in matrix analysis, there

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is an active area, where some interesting matrix or norm inequalities are derived from their scalar counterparts. Such inequalities may hold for operators acting on an infinite dimensional separable Hilbert space. This is based on the fact that self-adjoint operators (Hermitian matrices) can be regarded as a generalization of real numbers. A natural generalization of the classical Hermite–Hadamard inequality to Hermitian matrices could be the double inequality

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B) dt \leq \frac{f(A) + f(B)}{2}, \quad (1.3)$$

which is however not true, in general. To see this let us consider the convex function $f(t) = t^3$ and matrices $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then some straightforward computations show that $\left(\frac{A+B}{2}\right)^3 = \begin{pmatrix} 17/4 & 7/4 \\ 7/4 & 3/4 \end{pmatrix}$, $\int_0^1 (tA + (1-t)B)^3 dt = \begin{pmatrix} 31/6 & 5/2 \\ 5/2 & 4/3 \end{pmatrix}$, $\frac{A^3+B^3}{2} = \begin{pmatrix} 7 & 4 \\ 4 & 5/2 \end{pmatrix}$ and that not both inequalities of (1.3) simultaneously are true.

In this paper, we present some operator inequalities of Hermite–Hadamard type in which we use the convexity instead of the operator convexity. To do this, we first restrict ourselves to the monotone convex functions to get a majorization version as our main result. We then utilize the Mond–Pečarić method [17, 10, 21] to get another operator version of inequality (1.2). We also present some applications. Finally we generalize the main result of [8] for operator convex functions, positive linear maps and operators on (not necessarily finite dimensional) Hilbert space.

2. PRELIMINARIES

Let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and $I_{\mathcal{H}}$ is the identity operator. In the case where $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra \mathcal{M}_n of all $n \times n$ matrices with entries in the complex field \mathbb{C} . We denote by $\mathcal{H}_n(J)$ the set of all Hermitian matrices in \mathcal{M}_n , whose spectra are contained in an interval $J \subseteq \mathbb{R}$. By I_n we denote the identity matrix of \mathcal{M}_n . An operator $A \in \mathbb{B}(\mathcal{H})$ is called positive (positive-semidefinite for matrices) if $\langle A\xi, \xi \rangle \geq 0$ holds for every $\xi \in \mathcal{H}$ and then we write $A \geq 0$. In particular, if A is invertible and positive (positive-definite for matrices), then we write $A > 0$. For self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. A map Φ between C^* -algebras of operators is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. Throughout the paper all real-valued functions are assumed to be continuous. A real-valued function f defined on an interval J is called operator

convex if $f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathcal{H})$ with spectra in J and all $\lambda \in [0, 1]$. Of course, there are several equivalent version of the operator convexity in the literature, see [10, Chapter I] and [18] and references therein.

For a Hermitian matrix $A \in \mathcal{M}_n$, we denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ the eigenvalues of A arranged in the decreasing order with their multiplicities counted. The notation $\lambda(A)$ stands for the row vector $(\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A))$. The eigenvalue inequality $\lambda(A) \leq \lambda(B)$ means that $\lambda_j(A) \leq \lambda_j(B)$ for all $1 \leq j \leq n$. As a matter of fact for any two Hermitian matrices A, B the inequality $\lambda(A) \leq \lambda(B)$ holds if and only if $A \leq U^*BU$ for some unitary matrix U . The weak majorization $\lambda(A) \prec_w \lambda(B)$ means $\sum_{j=1}^k \lambda_j(A) \leq \sum_{j=1}^k \lambda_j(B)$ ($k = 1, 2, \dots, n$). It is known that three kinds of orders defined above satisfy $A \leq B \Rightarrow \lambda(A) \leq \lambda(B) \Rightarrow \lambda(A) \prec_w \lambda(B)$. A norm $|||\cdot|||$ on \mathcal{M}_n is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathcal{M}_n$ and all unitary matrices $U, V \in \mathcal{M}_n$. The Ky Fan norms, the Schatten p -norms and the operator norm provide significant families of unitarily invariant norms. The Ky Fan dominance theorem states that $\lambda(A) \prec_w \lambda(B)$ if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$. For more information on matrix analysis the reader is referred to [5].

3. OPERATOR HERMITE–HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS

We start this section by recalling two useful lemmas.

Lemma 3.1. [15, Lemma 2.4 and Remark 2.5] (see also [5, p. 281] and [2, Theorem 2.3]) *Let $A \in \mathcal{H}_n(J)$, f be a convex function defined on J , $x \in \mathbb{C}^m$ and $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_m$ be a positive linear map. If either (i) Φ is unital and $\|x\| = 1$ or (ii) $\|x\| \leq 1, 0 \in J, f(0) \leq 0$ and $0 < \Phi(I_n) \leq I_m$, then*

$$f(\langle \Phi(A)x, x \rangle) \leq \langle \Phi(f(A))x, x \rangle.$$

Lemma 3.2. [5, p. 67] *If $A \in \mathcal{H}_n$, then*

$$\sum_{j=1}^k \lambda_j(A) = \max \sum_{j=1}^k \langle Ax_j, x_j \rangle \quad (1 \leq k \leq n),$$

where the maximum is taken over all choices of orthonormal vectors $x_1, x_2, \dots, x_k \in \mathbb{C}^n$.

We are ready to give the operator version of the first inequality of the Hermite–Hadamard inequality.

Theorem 3.3. *Let $A, B \in \mathcal{H}_n(J)$, f be a convex function on J and Φ be a positive linear map from \mathcal{M}_n to \mathcal{M}_m . If either (i) Φ is unital or (ii) $0 \in J, f(0) \leq 0$ and $0 < \Phi(I_n) \leq I_m$, then*

$$\lambda \left(f \left(\frac{\Phi(A) + \Phi(B)}{2} \right) \right) \prec_w \lambda \left(\Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) \right).$$

Proof. Suppose that $\lambda_1, \dots, \lambda_m$ are the eigenvalues of $\frac{\Phi(A) + \Phi(B)}{2}$ with u_1, \dots, u_m as an orthonormal system of corresponding eigenvectors arranged such that $f(\lambda_1) \geq f(\lambda_2) \geq \dots \geq f(\lambda_m)$. We have

$$\begin{aligned} & \sum_{j=1}^k \lambda_j \left(f \left(\frac{\Phi(A) + \Phi(B)}{2} \right) \right) \\ &= \sum_{j=1}^k f \left(\left\langle \frac{\Phi(A) + \Phi(B)}{2} u_j, u_j \right\rangle \right) \quad (\text{by our assumption on } u_j) \\ &\leq \sum_{j=1}^k \int_0^1 f(\langle t\Phi(A) + (1-t)\Phi(B) u_j, u_j \rangle) dt \\ &\quad (\text{by the classical Hermite – Hadamard inequality}) \\ &= \sum_{j=1}^k \int_0^1 f(\langle \Phi(tA + (1-t)B) u_j, u_j \rangle) dt \quad (\text{by the linearity of } \Phi) \\ &\leq \sum_{j=1}^k \int_0^1 \langle \Phi(f(tA + (1-t)B)) u_j, u_j \rangle dt \quad (\text{by Lemma 3.1}) \\ &= \sum_{j=1}^k \left\langle \int_0^1 \Phi(f(tA + (1-t)B)) dt u_j, u_j \right\rangle \\ &\quad (\text{by the linearity and continuity of the inner product}) \\ &\leq \sum_{j=1}^k \lambda_j \left(\int_0^1 \Phi(f(tA + (1-t)B)) dt \right) \quad (\text{by Lemma 3.2}) \\ &= \sum_{j=1}^k \lambda_j \left(\Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) \right) \\ &\quad (\text{by the linearity and continuity of } \Phi). \end{aligned}$$

□

Now we get some operator versions of the second inequality of the Hermite–Hadamard inequality in two fashions. The first version is for monotone convex functions and the second version, which is weaker than the first one, is just for convex functions. To present the first version we would extend the following interesting result of Bourin to the positive linear maps.

Lemma 3.4. [6, Theorem 2.2] *Let $A_1, \dots, A_k \in \mathcal{H}_n([\omega, \Omega])$ and f be an increasing convex function defined on $[\omega, \Omega]$ containing the spectra of A_i , $i = 1, \dots, k$. If Z_1, \dots, Z_k are matrices with $\sum_{i=1}^k Z_i^* Z_i = I_n$, then there is a unitary matrix U such that $f\left(\sum_{i=1}^k Z_i^* A_i Z_i\right) \leq U \left(\sum_{i=1}^k Z_i^* f(A_i) Z_i\right) U^*$.*

Theorem 3.5. *Let $A_1, \dots, A_k \in \mathcal{H}_n([\omega, \Omega])$ and f be an increasing convex function defined on $[\omega, \Omega]$ containing the spectra of A_i , $i = 1, \dots, k$. If $\Phi_1, \dots, \Phi_k : \mathcal{M}_n \rightarrow \mathcal{M}_m$ are positive linear maps such either (i) $\sum_{i=1}^k \Phi_i(I_n) = I_m$ or (ii) $0 \in J$, $f(0) \leq 0$ and $\sum_{i=1}^k \Phi_i(I_n) \leq I_m$, then there is a unitary matrix U such that $f\left(\sum_{i=1}^k \Phi_i(A_i)\right) \leq U \sum_{i=1}^k \Phi_i(f(A_i)) U^*$.*

Proof. First let us prove Lemma 3.4 whenever $0 \in J$, $f(0) \leq 0$ and $\sum_{i=1}^k \Phi_i(I_n) \leq I_m$: Lemma 3.4 with $k = 1$ and $0 \in J$, $f(0) \leq 0$, $Z^* Z \leq I_n$ instead of $Z^* Z = I_n$ is still true. In fact, due to $I_n - Z^* Z \geq 0$, there is a matrix Y such that $Z^* Z + Y^* Y = I_n$. Using Lemma 3.4, we have

$$f(Z^* A Z) = f(Z^* A Z + Y^* 0 Y) \leq U(Z^* f(A) Z + Y^* f(0) Y) U^* \leq U^* Z^* f(A) Z U^*$$

for some unitary U . The general case now follows by considering Z to be the column vector (Z_1, \dots, Z_k) and A to be the diagonal matrix $A = \text{diag}(A_1, \dots, A_k)$.

Second assume that A is a Hermitian matrix and $\Psi : \mathcal{M}_n \rightarrow \mathcal{M}_m$ is a positive linear map. Using the spectral decomposition $A = \sum_j \lambda_j E_j$ of A , the fact that $\sum_j \sqrt{\Psi(E_j)} \sqrt{\Psi(E_j)} = \sum_j \Psi(E_j) = \Psi(I_n)$, Lemma 3.4 and the paragraph above we have

$$\begin{aligned} f(\Psi(A)) &= f\left(\sum_j \lambda_j \Psi(E_j)\right) = f\left(\sum_j \sqrt{\Psi(E_j)} \lambda_j \sqrt{\Psi(E_j)}\right) \\ &\leq U \left(\sum_j \sqrt{\Psi(E_j)} f(\lambda_j) \sqrt{\Psi(E_j)}\right) U^* = U \left(\sum_j f(\lambda_j) \Psi(E_j)\right) U^* \\ &= U \Psi\left(\sum_j f(\lambda_j) E_j\right) U^* = U \Psi(f(A)) U^* \end{aligned} \tag{3.1}$$

for some unitary U .

Next assume that $A_1, \dots, A_k \in \mathcal{H}_n([\omega, \Omega])$. Set

$$\Psi(\text{diag}(A_1, \dots, A_k)) = \sum_{i=1}^k \Phi_i(A_i).$$

Then Ψ is clearly a positive linear map. Hence there is a unitary U such that

$$\begin{aligned} f\left(\sum_{i=1}^k \Phi_i(A_i)\right) &= f(\Psi(\text{diag}(A_1, \dots, A_k))) \\ &\leq U\Psi(f(\text{diag}(A_1, \dots, A_k)))U^* \quad (\text{by (3.1)}) \\ &= U\Psi(\text{diag}(f(A_1), \dots, f(A_k)))U^* \quad (\text{by the functional calculus}) \\ &= U\sum_{i=1}^k \Phi_i(f(A_i))U^*. \end{aligned}$$

□

We are in a situation to give a matrix version of the second inequality of the Hermite–Hadamard inequality.

Theorem 3.6. *Let $A, B \in \mathcal{H}_n([\omega, \Omega])$, f be an increasing convex function on $[\omega, \Omega]$ and $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_m$ be a positive linear map such that either (i) it is unital or (ii) $0 \in J$, $f(0) \leq 0$ and $\Phi(I_n) \leq I_m$. Then*

$$\lambda\left(\int_0^1 f(\Phi(tA + (1-t)B))\right) \leq \lambda\left(\frac{\Phi(f(A)) + \Phi(f(B))}{2}\right). \quad (3.2)$$

Proof. It follows from Theorem 3.5 for $k = 2, A_1 = A, A_2 = B, \Phi_1 = t\Phi$ and $\Phi_2 = (1-t)\Phi$ that

$$f(t\Phi(A) + (1-t)\Phi(B)) \leq U[t\Phi(f(A)) + (1-t)\Phi(f(B))]U^*$$

for some unitary U . Hence

$$\begin{aligned} \int_0^1 f(\Phi(tA + (1-t)B)) &= \int_0^1 f(t\Phi(A) + (1-t)\Phi(B)) dt \\ &\leq \int_0^1 U[t\Phi(f(A)) + (1-t)\Phi(f(B))]U^* dt \\ &= U \int_0^1 t\Phi(f(A)) + (1-t)\Phi(f(B)) dt U^* \\ &= U \left[\frac{\Phi(f(A)) + \Phi(f(B))}{2} \right] U^*. \end{aligned}$$

Thus we get (3.2). □

Using Theorems 3.3 and 3.6 with $\Phi(A) = A$ we obtain that

Corollary 3.7. *If $A, B \in \mathcal{H}_n([\omega, \Omega])$ and f is an increasing convex function on $[\omega, \Omega]$, then*

$$\lambda \left(f \left(\frac{A+B}{2} \right) \right) \prec_w \lambda \left(\int_0^1 f(tA + (1-t)B) dt \right) \prec_w \lambda \left(\frac{f(A) + f(B)}{2} \right).$$

In particular,

$$\text{Tr} \left(f \left(\frac{A+B}{2} \right) \right) \leq \text{Tr} \left(\int_0^1 f(tA + (1-t)B) dt \right) \leq \text{Tr} \left(\frac{f(A) + f(B)}{2} \right).$$

Remark 3.8. One can not deduce any relationship between $\langle f(\Phi(A))x, x \rangle$ and $\langle (\Phi(f(A)))x, x \rangle$ from the known relations $f(\langle \Phi(A)x, x \rangle) \leq \langle f(\Phi(A))x, x \rangle$ and $f(\langle \Phi(A)x, x \rangle) \leq \langle \Phi(f(A))x, x \rangle$.

The fact that the function $f(t) = t^r$ is increasing and convex for $r > 1$ yields that

Corollary 3.9. *Let $r > 1$, $A, B \in \mathcal{H}_n([\omega, \Omega])$ and $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_m$ be a positive linear map such that either (i) it is unital or (ii) $0 \in J, f(0) \leq 0$ and $\Phi(I_n) \leq I_m$. Then*

$$\left\| \left(\frac{\Phi(A) + \Phi(B)}{2} \right)^r \right\| \leq \left\| \int_0^1 \Phi((tA + (1-t)B)^r) dt \right\|$$

and

$$\left\| \int_0^1 (t\Phi(A) + (1-t)\Phi(B))^r dt \right\| \leq \left\| \frac{\Phi(A^r) + \Phi(B^r)}{2} \right\|.$$

Now we use the Mond–Pečarić method [10] to get the second version of the second inequality of the Hermite–Hadamard inequality .

Theorem 3.10. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be self-adjoint operators with spectra in $[\omega, \Omega]$, $f : [\omega, \Omega] \rightarrow (0, \infty)$ be a convex function and $\Phi : \mathcal{M}_n \rightarrow \mathcal{M}_m$ be a positive linear map. Then*

$$\begin{aligned} & \Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) \\ & \leq \max \left\{ \frac{\Omega - t}{\Omega - \omega} \cdot \frac{f(\omega)}{f(t)} + \frac{t - \omega}{\Omega - \omega} \cdot \frac{f(\Omega)}{f(t)} : t \in [\omega, \Omega] \right\} \frac{f(\Phi(A)) + f(\Phi(B))}{2}. \end{aligned} \quad (3.3)$$

Proof. Let A, B be Hermitian operators with the spectra in $[\omega, \Omega]$. It follows from the convexity of f that

$$f(t) = f \left(\frac{\Omega - t}{\Omega - \omega} \cdot \omega + \frac{t - \omega}{\Omega - \omega} \cdot \Omega \right) \leq \frac{\Omega - t}{\Omega - \omega} f(\omega) + \frac{t - \omega}{\Omega - \omega} f(\Omega)$$

for all $t \in [\omega, \Omega]$. Applying the functional calculus we obtain

$$f(tA + (1-t)B) \leq \frac{\Omega - tA + (1-t)B}{\Omega - \omega} f(\omega) + \frac{tA + (1-t)B - \omega}{\Omega - \omega} f(\Omega).$$

So that

$$\begin{aligned} \Phi(f(tA + (1-t)B)) &\leq \frac{\Omega - t\Phi(A) + (1-t)\Phi(B)}{\Omega - \omega} f(\omega) \\ &\quad + \frac{t\Phi(A) + (1-t)\Phi(B) - \omega}{\Omega - \omega} f(\Omega), \end{aligned}$$

whence for each unit vector $x \in \mathcal{H}$, we get

$$\begin{aligned} \langle \Phi(f(tA + (1-t)B)) x, x \rangle &\leq \frac{\Omega - \langle (t\Phi(A) + (1-t)\Phi(B)) x, x \rangle}{\Omega - \omega} f(\omega) \\ &\quad + \frac{\langle (t\Phi(A) + (1-t)\Phi(B)) x, x \rangle - \omega}{\Omega - \omega} f(\Omega). \end{aligned}$$

So

$$\begin{aligned} \int_0^1 \langle \Phi(f(tA + (1-t)B)) x, x \rangle dt &\leq \frac{\Omega - \int_0^1 \langle (t\Phi(A) + (1-t)\Phi(B)) x, x \rangle dt}{\Omega - \omega} f(\omega) \\ &\quad + \frac{\int_0^1 \langle (t\Phi(A) + (1-t)\Phi(B)) x, x \rangle dt - \omega}{\Omega - \omega} f(\Omega). \end{aligned}$$

Hence

$$\begin{aligned} \left\langle \int_0^1 \Phi(f(tA + (1-t)B)) dt x, x \right\rangle &\leq \frac{\Omega - \left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle}{\Omega - \omega} f(\omega) \\ &\quad + \frac{\left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle - \omega}{\Omega - \omega} f(\Omega). \end{aligned}$$

Therefore

$$\begin{aligned} \left\langle \Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) x, x \right\rangle &\leq \frac{\Omega - \left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle}{\Omega - \omega} f(\omega) \\ &\quad + \frac{\left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle - \omega}{\Omega - \omega} f(\Omega). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\langle \Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) x, x \rangle}{f \left(\left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle \right)} &\leq \frac{1}{f \left(\left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle \right)} \\ &\times \left(\frac{\Omega - \left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle}{\Omega - \omega} f(\omega) + \frac{\left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle - \omega}{\Omega - \omega} f(\Omega) \right). \end{aligned}$$

Thus

$$\begin{aligned} &\left\langle \Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) x, x \right\rangle \\ &\leq \alpha f \left(\left\langle \frac{\Phi(A) + \Phi(B)}{2} x, x \right\rangle \right) \\ &\leq \alpha f \left(\frac{\langle \Phi(A)x, x \rangle + \langle \Phi(B)x, x \rangle}{2} \right) \\ &\leq \alpha \frac{f(\langle \Phi(A)x, x \rangle) + f(\langle \Phi(B)x, x \rangle)}{2} \quad (\text{by the convexity of } f) \\ &\leq \alpha \frac{\langle f(\Phi(A))x, x \rangle + \langle f(\Phi(B))x, x \rangle}{2} \quad (\text{by } f(\langle Ax, x \rangle) \leq \langle f(A)x, x \rangle) \\ &\leq \left\langle \alpha \frac{f(\Phi(A)) + f(\Phi(B))}{2} x, x \right\rangle, \end{aligned}$$

where $\alpha = \max \left\{ \frac{\Omega-t}{\Omega-\omega} \cdot \frac{f(\omega)}{f(t)} + \frac{t-\omega}{\Omega-\omega} \cdot \frac{f(\Omega)}{f(t)} : t \in [\omega, \Omega] \right\}$. Hence inequality (3.3) holds. \square

It follows from the Ky Fan Dominance Theorem (see [19]), Theorem 3.3 and Theorem 3.10 that

Corollary 3.11. *Let $A, B \in \mathcal{H}_n([\omega, \Omega])$, f be a convex function on $[\omega, \Omega]$ and Φ be a positive linear map from \mathcal{M}_n to \mathcal{M}_m . If either (i) Φ is unital or (ii) $0 \in [\omega, \Omega]$, $f(0) = 0$ and $0 < \Phi(I_n) \leq I_m$, then*

$$\begin{aligned} &\left\| \left\| f \left(\frac{\Phi(A) + \Phi(B)}{2} \right) \right\| \right\| \\ &\leq \left\| \left\| \Phi \left(\int_0^1 f(tA + (1-t)B) dt \right) \right\| \right\| \\ &\leq \max \left\{ \frac{\Omega-t}{\Omega-\omega} \cdot \frac{f(\omega)}{f(t)} + \frac{t-\omega}{\Omega-\omega} \cdot \frac{f(\Omega)}{f(t)} : t \in [\omega, \Omega] \right\} \left\| \left\| \frac{f(\Phi(A)) + f(\Phi(B))}{2} \right\| \right\|. \end{aligned}$$

The mapping $\Phi(A) = \sum_{i=1}^k X_i^* A X_i$ is a positive linear map. So that we infer the following result from Theorem 3.3 and Theorem 3.10.

Corollary 3.12. *Let $A, B \in \mathcal{H}_n([\omega, \Omega])$, f be a convex function on $[\omega, \Omega]$ and $X_1, \dots, X_k \in \mathcal{M}_n$ such that $\sum_{i=1}^k X_i^* X_i = I_n$. Then*

$$\begin{aligned} & \left\| \left\| f \left(\frac{1}{2} \sum_{i=1}^k X_i^* (A + B) X_i \right) \right\| \right\| \\ & \leq \left\| \left\| \sum_{i=1}^k X_i^* \int_0^1 f(tA + (1-t)B) dt X_i \right\| \right\| \\ & \leq \max \left\{ \frac{\Omega - t}{\Omega - \omega} \cdot \frac{f(\omega)}{f(t)} + \frac{t - \omega}{\Omega - \omega} \cdot \frac{f(\Omega)}{f(t)} : t \in [\omega, \Omega] \right\} \\ & \quad \times \left\| \left\| \frac{f(\sum_{i=1}^k X_i^* A X_i) + f(\sum_{i=1}^k X_i^* B X_i)}{2} \right\| \right\|. \end{aligned}$$

4. OPERATOR HERMITE–HADAMARD TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS

In this section we generalize the main result of [8].

Theorem 4.1. *If A, B are self-adjoint operators on a Hilbert space H with spectra in an interval J , f is an operator convex function on J and k, p are positive integers, then*

$$\begin{aligned} f \left(\frac{A + B}{2} \right) & \leq \frac{1}{k^p} \sum_{i=0}^{k^p-1} f \left(\frac{2i+1}{2k^p} A + \left(1 - \frac{2i+1}{2k^p} \right) B \right) \\ & \leq \int_0^1 f(tA + (1-t)B) dt \\ & \leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f \left(\frac{i+1}{k^p} A + \left(1 - \frac{i+1}{k^p} \right) B \right) + f \left(\frac{i}{k^p} A + \left(1 - \frac{i}{k^p} \right) B \right) \right] \\ & \leq \frac{f(A) + f(B)}{2}. \end{aligned} \tag{4.1}$$

Proof. Let $x \in H$ be a unit vector. It is easy to see that the function $\rho(t) = \langle f(tA + (1-t)B)x, x \rangle$ is a real-valued convex function on the interval $[0, 1]$, see [8, Theorem 2.1]. Utilizing the classical Hermite–Hadamard inequality on the interval $[\frac{i}{k^p}, \frac{i+1}{k^p}]$, we get that

$$\rho \left(\frac{2i+1}{2k^p} \right) \leq k^p \int_{\frac{i}{k^p}}^{\frac{i+1}{k^p}} \rho(t) dt \leq \frac{\rho \left(\frac{i}{k^p} \right) + \rho \left(\frac{i+1}{k^p} \right)}{2}.$$

Summation of the above inequalities over $i = 0, 1, \dots, k^p - 1$ yields

$$\sum_{i=0}^{k^p-1} \rho\left(\frac{2i+1}{2k^p}\right) \leq k^p \int_0^1 \rho(t) dt \leq \sum_{i=0}^{k^p-1} \frac{\rho\left(\frac{i}{k^p}\right) + \rho\left(\frac{i+1}{k^p}\right)}{2}.$$

Hence

$$\begin{aligned} & \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(\frac{2i+1}{2k^p}A + \left(1 - \frac{2i+1}{2k^p}\right)B\right) \\ & \leq \int_0^1 f(tA + (1-t)B) dt \\ & \leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(\frac{i+1}{k^p}A + \left(1 - \frac{i+1}{k^p}\right)B\right) + f\left(\frac{i}{k^p}A + \left(1 - \frac{i}{k^p}\right)B\right) \right]. \end{aligned} \quad (4.2)$$

By the operator convexity of f we have

$$\begin{aligned} & \frac{1}{k^p} \sum_{i=0}^{k^p-1} f\left(\frac{2i+1}{2k^p}A + \left(1 - \frac{2i+1}{2k^p}\right)B\right) \\ & \geq f\left[\frac{1}{k^p} \sum_{i=0}^{k^p-1} \left(\frac{2i+1}{2k^p}A + \left(1 - \frac{2i+1}{2k^p}\right)B\right)\right] \\ & = f\left[\frac{\sum_{i=0}^{k^p-1}(2i+1)}{2k^{2p}}A + \left(1 - \frac{\sum_{i=0}^{k^p-1}(2i+1)}{2k^{2p}}\right)B\right] \\ & = f\left(\frac{A+B}{2}\right) \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} & \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[f\left(\frac{i+1}{k^p}A + \left(1 - \frac{i+1}{k^p}\right)B\right) + f\left(\frac{i}{k^p}A + \left(1 - \frac{i}{k^p}\right)B\right) \right] \\ & \leq \frac{1}{2k^p} \sum_{i=0}^{k^p-1} \left[\frac{i+1}{k^p} f(A) + \left(1 - \frac{i+1}{k^p}\right) f(B) + \frac{i}{k^p} f(A) + \left(1 - \frac{i}{k^p}\right) f(B) \right] \\ & = \frac{f(A) + f(B)}{2}. \end{aligned} \quad (4.4)$$

Now (4.2), (4.3) and (4.4) yield the whole inequalities (4.1) as desired. \square

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